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On a Variant of a Theorem of Schmeidler

*M. Ali Khan*



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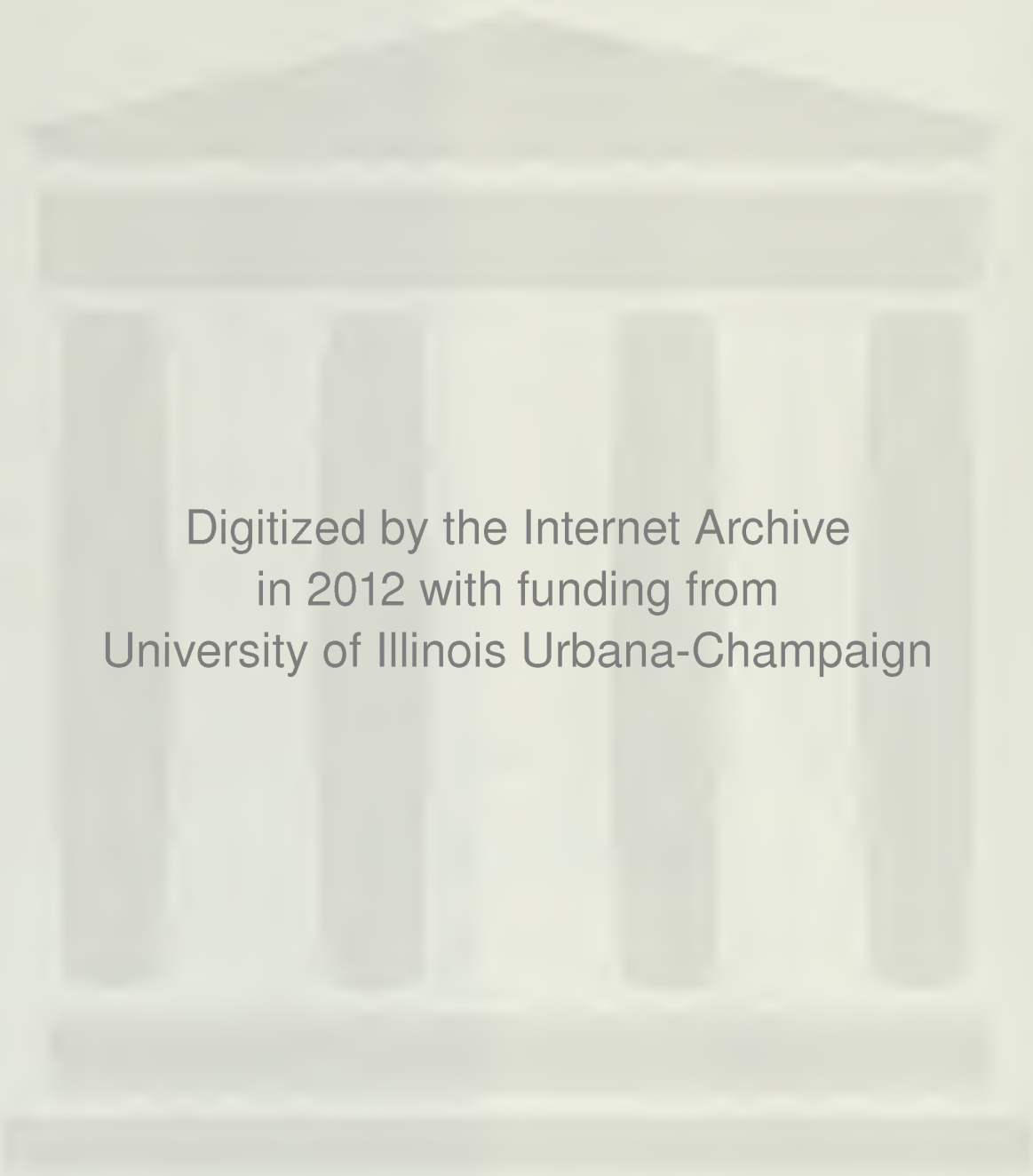
College of Commerce and Business Administration

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November 1986

On a Variant of a Theorem of Schmeidler

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On a Variant of a Theorem of Schmeidler†

by

M. Ali Khan\*

August 1986

Revised: November 1986

Abstract. We report generalizations of a theorem of Mas-Colell on the existence of a Cournot Nash equilibrium distribution for games in which the strategy space is not necessarily metrizable and separable and the pay-off functions are not necessarily continuous.

†I am grateful to Ye Neng Sun for finding a gap in the proof of Theorem 3.1 in the first version of this paper and to Tapan Mitra for several helpful discussions. Errors are, of course, solely mine. This research was presented at the Mathematical Economics Workshop at Indiana University and was supported in part by the Bureau of Business and Economic Research at the University of Illinois and in part by an NSF grant. Both sources of support are gratefully acknowledged.

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## 1. Introduction

Mas-Colell (1984) presents a reformulated version of Schmeidler's (1973) results on the existence of Cournot-Nash equilibria in games with a continuum of players. Taking his lead from Hart-Hildenbrand-Kohlberg (1974), Mas-Colell views a game as a probability measure  $\mu$  on the space of pay-off functions,  $\mathcal{U}_A$ , each such function being defined on the product of the given strategy or action space  $A$  and the space of probability measures on  $A$ ,  $\mathcal{M}$ . The dependence of a player's pay-off on the actions of the other players' actions is summarized by the dependence of the pay-off on  $\mathcal{M}$ . A Cournot-Nash equilibrium is then formulated as a measure  $\tau$  on the product space  $A \times \mathcal{U}_A$  such that

- (i) the marginal of  $\tau$  on  $\mathcal{U}_A$  is the game  $\mu$  itself,
- (ii) the maximizing action pay-off pairs, given  $\tau$ , have full  $\tau$  measure.

Under the assumption of  $A$  being a compact metric space and the pay-off functions being continuous, Mas-Colell shows the existence of a Cournot-Nash equilibrium as a consequence of the Ky Fan fixed point theorem and with the space of measures being endowed with the weak\* topology.

The hypothesis of a compact action space is natural enough but the assumption of it being metric is less easily justified. Indeed, in the context of a Banach space which is not necessarily separable, weakly compact action spaces are not necessarily metrizable. The same is true for weak\* compact sets of Banach spaces whose preduals are not separable. It is thus natural to ask whether Mas-Colell's theorem can be proved without the metric hypothesis. This is the first question we investigate in this note.



An answer to this question is of interest not only because it examines the robustness of Mas-Colell's conception but also because it leads naturally to the study of measures on a product space, one of whose elements is compact but not metric and the other is metrizable but not separable. In particular,  $A$  not being separable leads to  $\mathcal{U}_A$  not being separable [see, for example, Willard (1970, p. 282)]. Furthermore, it is far from clear how one could accommodate non-separable action spaces in the theory as originally laid out by Schmeidler and developed in Khan [1985] and subsequent references. This theory relies crucially on measurable selection theorems.

In the context of our first question, we present three results. First, we show that the metric assumption on  $A$  can be dispensed with in Mas-Colell's theorem if a game is defined as a Radon (tight) measure on the space of pay-off functions. Since every abstract measure on a complete, separable metric space is a Radon measure, our result is a generalization of that of Mas-Colell. Indeed, it is precisely the loss of the validity of this observation in the non-metrizable set-up that leads to complications of the argument. For example, Theorem 5.8 in Parthasarathy (1967, Chapter II) on the existence of a Radon measure is no longer available; nor Theorem 3.2 in Billingsley (1968) which characterizes weak convergence of a measure in terms of the weak convergence of its marginals [also see Parthasarathy (1967, Chapter III, Lemma 1.1)]. Nevertheless, one can appeal to results in Schwartz (1973) and Topsoe (1970) and provide a proof along the lines laid by Mas-Colell. These results also allow us to make an observation as regards

the closedness of the graph of the Cournot-Nash equilibrium correspondence. This constitutes the second result of the paper.

Mas-Colell also asked if there exists a Cournot-Nash equilibrium distribution  $\tau$  and a measurable function  $f$  from the space of pay-offs to the space of actions such that  $\tau$  gives full measure to the graph of  $f$ . He called such equilibria symmetric and showed their existence for finite actions, atomless games. In an analogous vein, our third result shows that there exist Cournot-Nash equilibria which can be viewed as a suitable integral of a measurable function from the space of pay-offs to measures on the action space. We call these disintegrated Cournot-Nash equilibria.

Since every upper semicontinuous function attains its maximum on a compact set, it is natural to ask if the above theory generalizes to upper semicontinuous, rather than continuous, pay-offs. This is the second question we investigate in this note. An answer to this question is of particular interest in light of recent emphasis on games with discontinuous pay-offs; see, for example, Dasgupta-Maskin (1986) and their references.

The principal technical difficulty with such an extension lies in the fact that the sup norm topology is no longer available and one needs to formulate a topology on the space of pay-offs that is able to fulfill the demands made on it in the course of the proofs. Such a topology is available from the recent work of Dolecki-Salinetti-Wets (1983) and is simply motivated by the observation that every lower semicontinuous function has a closed epigraph. There are, of course,

several topologies on the space of closed subsets of a topological space and Dolecki et al. (1983) choose the topology of closed convergence. This topology is, by now, well understood in mathematical economics; see, for example, Hildenbrand (1974).

The plan of the paper is as follows. Section 2 presents some preliminary results on the space of Radon measures on a compact Hausdorff space. These results are well known but are not found in standard texts such as Billingsley (1968), Hildenbrand (1974) or Parthasarathy (1967). Section 3 develops the theory with continuous pay-offs and Section 4 with upper semicontinuous ones. It should be possible to proceed from the general case of Section 4 to the particular but relaxing one assumption at a time is, in our opinion, more instructive and does not lead to undue repetition. Section 5 is devoted to the proofs.

## 2. Mathematical Preliminaries

In this section we collect some results on the space of Radon probability measures on topological spaces that are not necessarily separable or metrizable.

Let  $X$  be a Hausdorff topological space and  $\mathcal{M}_+(X)$  the space of all non-negative, finite measures defined on  $\mathcal{B}(X)$ , the Borel  $\sigma$ -algebra on  $X$ .  $\mu \in \mathcal{M}_+(X)$  is said to be a Radon measure or tight if

$$\mu(B) = \sup\{\mu(K) : K \text{ compact, } K \subset B\} \text{ for all } B \in \mathcal{B}(X)$$

We shall denote the set of all Radon measures on  $X$  by  $\mathcal{M}_+(X, t)$  and the set of all probability measures and Radon probability measures by  $\mathcal{M}_+^1(X)$  and  $\mathcal{M}_+^1(X, t)$ , respectively.



Theorem 2.1. If  $A \in \mathcal{B}(X)$  and  $\mu \in \mathcal{M}_+(X, \tau)$ , then  $\mu|_A$ , the restriction of  $\mu$  to  $A$  is a measure in  $\mathcal{M}_+(A, \tau)$ .

Proof: See p. 14 in Topsoe (1970).

Following Topsoe (1970), Schwartz (1973) and others, we define the weak topology on  $\mathcal{M}_+(X)$  as the weakest topology on  $\mathcal{M}_+(X)$  for which every map  $\mu \mapsto \mu(f)$ , where  $f: X \rightarrow \mathbb{R}$  is bounded and upper semi-continuous, is upper semi-continuous. Note that

$$\mu(f) = \int_X f(x) d\mu(x)$$

The induced topology on  $\mathcal{M}_+(X, \tau)$  and  $\mathcal{M}_+^1(X, \tau)$  will also be called the weak topology and these spaces are always assumed to be endowed with this topology.

Theorem 2.2.  $\mathcal{M}_+(X, \tau)$  is Hausdorff for any Hausdorff topological space  $X$ .

Proof: See Topsoe (1970, Theorem 11.2) or Schwartz (1973, Proposition 2, p. 371).

Theorem 2.3.  $\mathcal{M}_+^1(X, \tau)$  is compact if and only if  $X$  is compact.

Proof: See, for example, the Notes to Section 11 in Topsoe (1970, p. 76). Also Schwartz (1973, p. 379).

Theorem 2.4. Let  $\{\gamma^\alpha\}$  be a net in  $\mathcal{M}_+(X)$  and  $\gamma$  an element of  $\mathcal{M}_+(X)$ . Then  $\gamma^\alpha$  converges to  $\gamma$  if and only if

- (i)  $\limsup \gamma^\alpha(F) \leq \gamma(F)$  for all closed subsets F of X,  
(ii)  $\limsup \gamma^\alpha(X) = \gamma(X).$

Proof: See Theorem 8.1 of Topsoe (1970, p. 40).

Our next two results relate convergence of a net of measures to convergence of their marginals. For a measure  $\rho$  on a product space  $S \times T$ , let  $\rho_S, \rho_T$  be the marginal measures on  $S$  and  $T$  respectively.

Theorem 2.5. Let  $\{\gamma^\alpha\}$  be a net in  $\mathcal{M}_+^1(S \times T, t)$  which converges to  $\gamma$  in  $\mathcal{M}_+^1(S \times T, t)$ . Then  $\gamma_i^\alpha$  converges to  $\gamma_i$ ,  $i = S, T$ .

Proof: Without loss of generality, let  $i = S$  and pick an arbitrary closed subset  $F$  of  $S$ . Then  $(F \times T)$  is closed and we appeal to Theorem 2.4 to finish the proof.

Theorem 2.6. Let  $S$  and  $T$  be completely regular spaces and  $\{\gamma^\alpha\}$  be a net in  $\mathcal{M}_+(S \times T, t)$ . If

$$\gamma_S^\alpha \rightarrow \mu, \gamma_T^\alpha \rightarrow \nu$$

for some  $\mu \in \mathcal{M}_+(S, t)$  and some  $\nu \in \mathcal{M}_+(T, t)$ , then  $\{\gamma^\alpha\}$  has a limit point  $\gamma \in \mathcal{M}_+(S \times T, t)$  with marginals  $\mu$  and  $\nu$ .

Proof: See the proof of Lemma 5.1 in Hoffman-Jorgenson (1970).

Our next result shows that Borel probability measures with finite support are dense in  $\mathcal{M}_+^1(X)$ .

Theorem 2.7. Let  $\mathcal{E}$  be the set of Dirac point measures on  $(X, \mathcal{B}(X))$ . Then the closed convex hull of  $\mathcal{E}$  is  $\mathcal{M}_+^1(X)$ .

Proof: See the proof of Theorem 11.1 in Topsoe (1970, p. 48).

Next, we present a general result on the disintegration of measures. For any two measurable spaces  $(F, \mathcal{F})$ ,  $(G, \mathcal{G})$ , we shall denote the product  $\sigma$ -algebra on  $F \times G$  by  $\mathcal{F} \times \mathcal{G}$ . Furthermore, for any  $W \in \mathcal{F} \times \mathcal{G}$ , and any  $f \in F$ , let  $W_f = \{g \in G: (f, g) \in W\}$ . The sections of  $W$  on  $G$  are defined similarly.

Theorem 2.8. Let  $X$  be a Hausdorff topological space,  $(S, \mathcal{S})$  a measurable space,  $\tau$  a probability measure on  $(S \times X, \mathcal{S} \times \mathcal{B}(X))$ , and the marginal of  $\tau$  on  $S$ ,  $\tau_S$ , be Radon. Then there exists a family of probability measures on  $X$ ,  $(\rho_s)_{s \in S}$ ,  $\rho_s \in \mathcal{M}_+^1(X)$ , such that for any  $W \in \mathcal{F} \times \mathcal{B}(X)$ ,

- (i)  $h_W: S \rightarrow \mathbb{R}$ ,  $h_W(s) = \rho_s(W_s)$ , is measurable,
- (ii)  $\tau(W) = \int_S h_W(s) d\tau_S(s)$

Proof: See the proof of Theorem 3.1 in Edgar (1975).

Finally, for the record, we state

Theorem 2.9. Let  $\mu$  be a Borel probability measure on a topological space  $(X, \tau)$ . If  $\tau$  is metrizable with a complete metric and is separable, then  $\mu \in \mathcal{M}_+^1(X, \tau)$ .

Proof: See Billingsley (1968, Theorem 1.4) or Parthasarathy (1967, II, Theorem 3.2). More generally, see Topsoe (1970, p. 16).



### 3. The Model and Results with Continuous Pay-offs

Let  $A$  be a nonempty, compact Hausdorff space of actions. A player is characterized by a continuous utility function  $u: A \times \mathcal{M}_+^1(A, t) \rightarrow \mathbb{R}$ . Let  $\mathcal{U}_A$  be the space of continuous utility functions endowed with the supremum norm topology; recall that  $\mathcal{M}_+^1(A, t)$  is compact by virtue of Theorem 2.2. We can now state

Definition 3.1. A game is a Borel probability measure  $\mu$  on  $\mathcal{U}_A$ .

Definition 3.2. A Borel probability measure  $\tau$  on  $\mathcal{U}_A \times A$  is a Cournot-Nash equilibrium distribution of a game  $\mu$  if

- (i)  $\tau \mathcal{U}_A = \mu$ ,
- (ii)  $\tau(\{(u, a): u(a, \tau_A) \geq u(A, \tau_A)\}) \equiv \tau(B_\tau) = 1$ .

We can now present our result.

Theorem 3.1. If  $\mu \in \mathcal{M}_+^1(\mathcal{U}_A, t)$ , there exists a Cournot-Nash equilibrium distribution for the game  $\mu$ .

Remark. The proof shows that the Cournot-Nash equilibrium distribution is Radon, i.e., an element of  $\mathcal{M}_+^1(A \times \mathcal{U}_A, t)$ .

Corollary (Mas-Colell): If  $A$  is metrizable, then the theorem is true for any Borel probability measure on  $\mathcal{U}_A$ .

We also present a result on the upper hemicontinuity of the Cournot-Nash correspondence. For any game  $\mu$ , let  $\Gamma(\mu)$  denote its set of Cournot-Nash equilibria.

Theorem 3.2.  $\Gamma: \mathcal{M}_+^1(\mathcal{U}_A, t) \rightarrow \mathcal{M}_+^1(A \times \mathcal{U}_A, t)$  has a closed graph.

Our final result requires a definition.

Definition 3.3. A Cournot-Nash equilibrium distribution  $\tau$  can be disintegrated if there exist a family of probability measures,  $\rho_u$ ,  $u \in \mathcal{U}_A$ ,  $\rho_u \in \mathcal{M}_+^1(\mathcal{U}_A)$ , such that for any  $W \in \mathcal{B}(\mathcal{U}_A) \times \mathcal{B}(A)$ ,

(i)  $h_W: \mathcal{U}_A \rightarrow \mathbb{R}$ ,  $h_W(u) = \rho_u(W_u)$ , is measurable,

(ii)  $\tau(W) = \int_A \rho_u(W_u) d\mu(u)$

We can now present

Theorem 3.3. If  $\mu \in \mathcal{M}_+^1(\mathcal{U}_A, t)$ , then every element of  $\Gamma(\mu)$  can be disintegrated. Hence, there exists a disintegrated Cournot-Nash equilibrium distribution for the game  $\mu$ .

We finish this subsection by asking whether Theorems 3.1 to 3.3 are true for games which are not necessarily in  $\mathcal{M}_+^1(\mathcal{U}_A, t)$  but are in  $\mathcal{M}_+^1(\mathcal{U}_A)$ , i.e., for games which are not necessarily Radon but are Borel probability measures. One may make two observations in this context. First, there certainly exist Borel probability measures even on compact Hausdorff spaces that are not Radon. The reader can see, for example, Dieudonné's construction [Schwartz (1973); p. 45]. Second, and on the other hand, Edgar (1975, p. 448) makes the observation that "it is consistent with the usual axioms of set theory (indeed, it follows from the Axiom of Constructibility) that if  $X$  is a complete metric space, then every element of  $\mathcal{M}_+(X)$  belongs to  $\mathcal{M}_+^1(X, t)$ , i.e., every Borel probability measure is tight." The reader need only be

reminded that  $\mathcal{U}_A$  is a complete metric space [see, for example, Willard (1970), Theorem 42.10].

#### 4. The Model and Results: Upper Semicontinuous Pay-offs

In this section we continue to assume that  $A$  is a nonempty, compact Hausdorff space of actions but characterize a player by an upper semicontinuous utility function  $u: \text{Ax}^1_+(A, t) \rightarrow \bar{\mathbb{R}}$ , where  $\bar{\mathbb{R}}$  is the space of extended reals.

Following Dolecki-Salinetti-Wets (1983), we endow the space of upper semicontinuous utility functions with the hypo-topology and denote the resulting space by  $\mathcal{U}_A^h$ . Two elements of  $\mathcal{U}_A^h$  are "close" in the hypo-topology if their hypographs are "close" in the topology of closed convergence. Recall that the hypograph of a function  $f: X \rightarrow \bar{\mathbb{R}}$  is given by the set  $\{(x, \eta) \in X \times \mathbb{R}: f(x) \geq \eta\}$ . Since a function is upper semicontinuous if and only if its hypograph is a closed set, we have a well defined topology.

The following result on the space  $\mathcal{U}_A^h$  is fundamental for this section.

Theorem 4.1.  $\mathcal{U}_A^h$  is Hausdorff compact.

Armed with Theorem 4.1, we can now state a general proposition.

Theorem 4.2. Theorems 3.1 to 3.3 are valid with  $\mathcal{U}_A^h$  substituted for  $\mathcal{U}_A$ . Moreover, Theorem 3.2 can be strengthened to say that  $\Gamma$  is an upper hemicontinuous mapping.



It is clear that Theorem 4.2 generalizes the theory developed in Section 3 to a larger class of pay-off functions and hence to a larger class of games. However, if we limit ourselves to the space of continuous functions, it is natural to ask whether the results in Theorem 4.2 are stronger than the corresponding ones in Section 3. Put another way, what we are asking is whether the hypo-topology is finer than the sup-norm topology on the space of continuous pay-offs. The answer is no as can easily be seen by considering a sequence of real-valued functions  $\{f^n\}_{n=1}^{\infty}$ ,  $f^n: [0,1] \rightarrow [0,1]$  where

$$\begin{aligned} f^n(x) &= nx & 0 \leq x < 1/n \\ &= 1 & 1/n \leq x \leq 1 \end{aligned}$$

Let  $f^{\infty}(x) = 1$  for all  $0 \leq x \leq 1$ . Then  $f^n \rightarrow f^{\infty}$  in the hypo-topology but not in the sup-norm topology. It is also easy to check that the sup-norm topology is finer than the hypo-topology, i.e., a net  $\{f^n\}$  converges to  $f$  in the hypo-topology if it converges in the sup-norm topology.

However, since the negative of every lower semicontinuous function is upper semicontinuous, the space of lower semicontinuous functions can be endowed with the "negative" of the hypo-topology, the epi-topology. Furthermore, since the space of continuous utility functions is both upper and lower semicontinuous, we can endow this space with the smallest topology containing both the epi- and hypo-topologies. Dolecki et al. (1983) term this the  $\pm e$  topology. It is now natural to ask whether the theory developed in Section 3 extends to

$\mathcal{U}_A^{\pm e}$ , the space of continuous pay-off functions endowed with the  $\pm e$  topology, and whether this extension generalizes the results of Section 3. We can now offer

Theorem 4.3. Theorems 3.1 to 3.3 are valid with  $\mathcal{U}_A^{\pm e}$  substituted for  $\mathcal{U}_A$ . This substitution also yields a generalization of Theorems 3.1 to 3.3.

## 5. Proofs

### Proof of Theorem 1

As in Mas-Colell (1984), the proof is an application of the Ky Fan fixed point theorem [see, for example, Berge (1963, p. 251)]. We show in a series of claims that all the conditions for the applicability of this theorem are satisfied. Let

$$\mathcal{I} = \{\tau \in \mathcal{M}_+^1(\mathcal{U}_A \times A, t) : \tau u_A = \mu\}.$$

Claim 1:  $\mathcal{I}$  is nonempty.

Since  $A$  is compact, the Riesz representation theorem [see, for example, Berberian (1965, Theorem 3, p. 120)] guarantees us that  $\mathcal{M}_+^1(A, t) \neq \emptyset$ . Since  $\mu \in \mathcal{M}_+^1(\mathcal{U}_A, t)$ , we can now appeal to Theorem 17 in Schwartz (1973, p. 63) to assert the existence of a unique Radon measure  $\lambda$  on  $A \times \mathcal{U}_A$  such that

$$(*) \quad \lambda(B \times C) = v(B)\mu(C) \text{ for all } B \in \mathcal{B}(A), C \in \mathcal{B}(\mathcal{U}_A).$$

Note that Schwartz states (\*) in his theorem in terms of the essential outer measure but since we are dealing with probability measures, the

distinction can be neglected. The reason for this is that we have defined a Radon measure in terms of Definition  $R_3$  of Schwartz (1973, p. 13) and from the proof of  $R_3 \Rightarrow R_1$  [Schwartz (1973, p. 13)], we see that the measure of a set with finite measure equals the essential outer measure of that set.

From (\*) we obtain

$$\lambda \mu_A(B) = \lambda(A \times B) = \nu(A)\mu(B) = \mu(B) \text{ for all } B \in \mathcal{B}(\mathcal{U}_A)$$

Since  $\lambda(A \times \mathcal{U}_A) = \nu(A)\mu(\mathcal{U}_A) = 1$  and since, by definition, measures are non-negative,  $\lambda \in \mathcal{I}$  and the proof of the claim is complete.

Claim 2.  $\mathcal{I}$  is convex.

Pick  $\mu^1, \mu^2$  from  $\mathcal{I}$  and  $\lambda$  a real number such that  $0 < \lambda < 1$ . Then it is routinely checked that the marginal of  $\lambda\mu^1 + (1-\lambda)\mu^2$  on  $\mathcal{U}_A$  is  $\mu$  and that  $\lambda\mu^1 + (1-\lambda)\mu^2 \in \mathcal{M}_+^1(\mathcal{U}_A \times A)$ .

Claim 3.  $\mathcal{I}$  is compact.

Pick a net  $\{\gamma^\alpha\}$  from  $\mathcal{I}$ . We first show that the marginal of  $\gamma^\alpha$  on  $A$ ,  $\gamma_A^\alpha$ , is an element of  $\mathcal{M}_+^1(A, \tau)$ . Let  $p_A$  be the projection map from  $A \times \mathcal{U}_A$  to  $A$ . Then  $\gamma_A^\alpha$  can be written as the image measure  $p_A \cdot \gamma^\alpha = \gamma^\alpha \cdot p_A^{-1}$ . Since  $p_A$  is a continuous function, and we are working with probability measures, the image measure is Radon [Schwartz (1973), last paragraph on p. 36)].

Since  $\{\gamma_A^\alpha\}$  is a net setting in  $\mathcal{M}_+^1(A, \tau)$ , we can appeal to Theorem 2.2 to assert the existence of a subnet, indicated by  $\{\gamma_A^\beta\}$ , such that

$$\gamma_A^\beta \rightarrow \gamma_A$$



and  $\gamma_A \in \mathcal{M}_+^1(A, t)$ . Since  $A$  is compact,  $A$  is completely regular [see, for example, Willard (1970, p. 95, Corollary 15.7, Theorem 17.10)]. Again, since  $\mathcal{U}_A$  is metric, it is completely regular [see, for example, Willard (1970, Example 14.9)]. We can now appeal to Theorem 2.6, to assert the existence of a measure  $\gamma \in \mathcal{M}_+^1(Ax \mathcal{U}_A)$  such that

- (i)  $\gamma|_{\mathcal{U}_A} = \mu$
- (ii)  $\gamma^\beta \rightarrow \gamma$ .

This completes the proof of the claim.

Claim 4.  $B: \mathcal{T} \rightarrow 2^{Ax \mathcal{U}_A}$  has a closed graph.

Let  $\tau^v \rightarrow \tau$ ,  $(u^v, a^v) \rightarrow (u, a)$  and  $(u^v, a^v) \in B_{\tau^v}$ . Suppose  $(u, a) \notin B_\tau$ . Then there exists  $\hat{x} \in A$  such that

$$u(\hat{x}, \tau_A) > u(a, \tau_A)$$

Hence there exists  $\varepsilon > 0$  such that

$$(\dagger) \quad u(\hat{x}, \tau_A) > u(a, \tau_A) + \varepsilon$$

Since  $\tau^v \rightarrow \tau$ , Theorem 2.5 yields that  $\tau_A^v \rightarrow \tau_A$ , and since  $u$  is jointly continuous on  $Ax \mathcal{M}_+^1(A, t)$ , there exists  $\bar{v}$  such that

$$|u(\hat{x}, \tau_A) - u(\hat{x}, \tau_A^v)| < \varepsilon/4 \text{ for all } v > \bar{v}.$$

By the same reasoning, there exists  $\bar{\bar{v}}$  such that

$$|u(a, \tau_A) - u(a^v, \tau_A^v)| < \varepsilon/4 \text{ for all } v > \bar{\bar{v}}.$$

This implies that

$$u(\hat{x}, \tau_A^v) > u(\hat{x}, \tau_A) - \varepsilon/4$$

$$u(a, \tau_A) > u(a^v, \tau_A^v) - \varepsilon/4$$

On substituting in † we obtain

$$u(\hat{x}, \tau_A^v) + \varepsilon/4 > u(a^v, \tau_A^v) + \varepsilon - \varepsilon/4 \text{ for all } v > \text{Max}(\bar{v}, \bar{\bar{v}}) \equiv v'.$$

This can be simplified to yield

$$(\dagger\dagger) \quad u(\hat{x}, \tau_A^v) > u(a^v, \tau_A^v) + \varepsilon/2 \text{ for all } v > v'$$

Since  $\mathcal{C}_A$  is endowed with the sup norm topology, there exists  $v''$  such that for all  $z \in A \times \mathcal{M}_+^1(A, t)$ ,

$$|u^v(z) - u(z)| < \varepsilon/4 \text{ for all } v > v''$$

Let  $v_0 > \text{Max}(v', v'')$  and deduce

$$u^{v_0}(\hat{x}, \tau_A^{v_0}) > u(\hat{x}, \tau_A^{v_0}) - \varepsilon/4$$

$$u(a^{v_0}, \tau_A^{v_0}) > u^{v_0}(a^{v_0}, \tau_A^{v_0}) - \varepsilon/4$$

On substituting in (††), we obtain

$$u^{v_0}(\hat{x}, \tau^{v_0}) + \varepsilon/4 > u^{v_0}(a^{v_0}, \tau^{v_0}) - \varepsilon/4 + \varepsilon/2$$

This simplifies to yield a contradiction to the fact that  $(u^{v_0}, a^{v_0}) \in$

$B_{\tau^{v_0}}$ . This contradiction proves the claim.

Claim 5. For any  $\tau \in \mathcal{J}$ ,  $B_\tau$  is a closed set in  $\mathcal{U}_A \times A$ .

This is a simple consequence of Claim 4.

Next, we consider the map  $Q: \mathcal{J} \rightarrow 2^{\mathcal{J}}$  such that

$$Q(\tau) = \{\rho \in \mathcal{J}: \rho(B_\tau) = 1\}$$

Claim 6. For any  $\tau \in \mathcal{J}$ ,  $Q(\tau) \neq \emptyset$ .

From Theorem 2.7, we know that there exists a net  $\{\mu^\nu\}$  converging to  $\mu$  and such that each  $\mu^\nu$  has a finite support. Pick a particular  $\mu^\nu$  and assume that its support consists of  $k$  elements,  $u_1, \dots, u_k$ . Let

$$S(u_i) = \{a \in A: u_i(a, \tau_A) \geq u_i(A, \tau_A)\}$$

Certainly,  $S(u_i)$  is a closed, and hence compact, subset of  $A$ . As in Claim 1, the Riesz representation theorem guarantees us an element  $\bar{v}_i$  in  $\mathcal{M}_+^1(S(u_i), t)$ . We can extend  $\bar{v}_i$  to  $\mathcal{B}(A)$  by defining for each  $W \in \mathcal{B}(A)$ ,

$$v_i(W) = \bar{v}_i(S(u_i) \cap W)$$

Since  $S(u_i)$  is a closed subset of  $A$ , certainly  $(S(u_i) \cap W) \in \mathcal{B}(S(u_i))$  and hence  $v_i$  is well-defined. [See, for example, Berberian (1965); Exercise 17 on p. 183]. It can also be easily checked that  $v_i$  is a measure. Certainly  $v_i(A) = \bar{v}_i(S(u_i) \cap A) = \bar{v}_i(S(u_i)) = 1$ . Furthermore, for any disjoint countable family  $W_i$  chosen from  $\mathcal{B}(A)$ ,  $v_i(\bigcup_i W_i) = (\bar{v}_i(S(u_i) \cap \bigcup_i W_i) = \bar{v}_i(\bigcup_i (S(u_i) \cap W_i) = \sum_i \bar{v}_i(S(u_i) \cap W_i) = \sum_i v_i(W_i)$ . Finally, to show that  $\bar{v}_i$  is Radon, pick any  $W_i \in \mathcal{B}(A)$  and such that  $v_i(W_i) > 0$ . This means that  $\bar{v}_i(S(u_i) \cap W_i) > 0$ . But  $\bar{v}_i$  is Radon and hence for any positive  $\varepsilon$ , there exists a compact set  $K \subseteq (S(u_i) \cap W_i)$

such that  $\bar{v}_i(K) > \bar{v}_i(S(u_i) \cap W_i) - \varepsilon$ . This implies that  $v_i(K) > v_i(W_i) - \varepsilon$  and we are done.

Now for any  $W \in \mathcal{B}(A) \times \mathcal{B}(\mathcal{U}_A)$  define

$$\rho^v(W) = \sum_{i=1}^k v_i(W_{u_i}) \mu^v(u_i)$$

where  $W_{u_i} = \{a \in A : (a, u_i) \in W\}$ . Certainly the sections  $W_{u_i}$  are measurable [see, for example, Berberian (1965); Theorem 3, p. 120]

and hence  $\rho^v$  is well defined. In order to show that  $\rho^v$  is a measure, pick a countable family of disjoint subsets  $W_j$  from  $\mathcal{B}(A) \times \mathcal{B}(\mathcal{U}_A)$ .

The second equality in the following demonstration follows from Theorem 2 on p. 120 in Berberian (1965); the third equality relies on  $v_i$  being measures and on  $(W_j)_{u_i}$  and  $(W_k)_{u_i}$ ,  $j \neq k$ , being disjoint.

$$\begin{aligned} \rho^v\left(\bigcup_j W_j\right) &= \sum_{i=1}^k v_i\left(\left(\bigcup_j W_j\right)_{u_i}\right) \mu^v(u_i) \\ &= \sum_{i=1}^k v_i\left(\bigcup_j (W_j)_{u_i}\right) \mu^v(u_i) \\ &= \sum_{i=1}^k \left(\sum_{j=1}^{\infty} v_i(W_j)_{u_i}\right) \mu^v(u_i) \\ &= \sum_{j=1}^{\infty} \rho^v(W_j) \end{aligned}$$

Next, in order to show that  $\rho^v$  is Radon, pick  $W \in \mathcal{B}(A) \times \mathcal{B}(\mathcal{U}_A)$  such that  $\rho^v > 0$ . Now for any  $\varepsilon > 0$  we have to find a compact set  $K \subseteq W$  such that  $\rho^v(K) > \rho^v(W) - \varepsilon$ . Since  $\sum_{i=1}^k v_i(W_{u_i}) \mu^v(u_i) > 0$ , without any loss of generality we can assume  $v_i(W_{u_i}) > 0$  for all  $i$ . Since  $v_i$  are Radon, we can find compact  $K_i \subseteq W_{u_i}$  such that



$$v_i(K_i) > v_i(W_{u_i}) - \varepsilon$$

Let  $K = \bigcup_{i=1}^k K_i$ . Certainly  $K$  is compact. Since  $K_i \cap K_j = \emptyset$  for  $i \neq j$ ,

$$\rho^v(K) = \sum_{i=1}^k v_i(K_i) \mu^v(u_i) > \rho^v(W) - \varepsilon.$$

Finally, we check that  $\rho_{\mathcal{U}_A}^v = \mu^v$ . This is easy since for any  $W \in \mathcal{B}(\mathcal{U}_A)$ ,  
 $\rho_{\mathcal{U}_A}^v(W) = \rho^v(AxW) = \sum_{i \in I} v_i(A) \mu^v(u_i) = \mu^v(W)$  where  $I = \{i \in (1, 2, \dots, k) : u_i \in W\}$ .

Since  $\mathcal{M}_+^1(A, t)$  is compact by virtue of Theorem 2.3, there exists a convergent subnet  $\rho_A^\alpha$  with limit  $\gamma_A$ . By construction we know that  $\rho_{\mathcal{U}_A}^\alpha = \mu^\alpha$ , and by hypothesis,  $\mu^\alpha$  has limit  $\mu$ . We now appeal to Theorem 2.6 to assert the existence of a limit point  $\rho$  of  $\rho^\alpha$  in  $\mathcal{M}_+^1(Ax\mathcal{U}_A, t)$  such that  $\rho_A = \gamma_A$  and  $\rho_{\mathcal{U}_A} = \mu$ . All that remains to be shown is that  $\rho(B_\tau) = 1$ . But by Theorem 2.4,

$$\rho(B_\tau) \geq \limsup \rho^\alpha(B_\tau) = 1.$$

Since  $\rho(Ax\mathcal{U}_A) = \lim \rho^\alpha(Ax\mathcal{U}_A) = 1$ , we are done.

Remark. In the case when  $A$  is compact metric, and hence separable, one has an alternative proof of Claim 6 that does not hinge on Theorem 2.7. Consider  $\Phi(u) = \{a \in A : (a, u) \in B_\tau\}$ , the  $u$ -section of  $B_\tau$  on  $A$ . Certainly  $\Phi(u) \neq \emptyset$  for all  $u \in \mathcal{U}_A$  and given Claim 5,  $\Phi$  has a measurable graph. Hence by Aumann's selection theorem [see Castaing-Valadier (1977), Theorem 3.22] a measurable selection  $h: \mathcal{U}_A \rightarrow A$ . Since  $A$  is separable metric,  $h$  is Lusin measurable, see Schwartz

(1973, Theorem 5, p. 26). Let  $f: \mathcal{U}_A \rightarrow A \times \mathcal{U}_A$  be defined by  $f(u) = (h(u), u)$ . Since  $\mu$  is a finite probability measure, certainly  $f$  is  $\mu$ -proper (Schwartz (1973, Definition 10, p. 31)) and hence the image measure  $f\mu$  is Radon (Schwartz (1973, p. 32)). It is now easy to check that  $f\mu(B_\tau) = 1$  and that  $(f\mu)_{\mathcal{U}_A}$ , the marginal of  $f\mu$  on  $\mathcal{U}_A$ , is  $\mu$ . Hence  $f\mu \in \mathcal{J}$  and we are done. This proof does not work in our generalized set-up because of the difficulty of finding Lusin measurable selections.

Claim 7. For any  $\tau \in \mathcal{J}$ ,  $Q(\tau)$  is convex.

This is straightforward.

Before considering our next claim, we develop some lemmata. For the definition and properties of the lim sup of a sequence of sets, see, for example, Klein-Thompson [1984].

Lemma 1. Let  $\{A_\nu\}$  be a net of subsets of a compact set  $K$  and such that  $\limsup A_\nu \subset W$  for an open set  $W$ . Then for all  $\bar{\nu}$ , there exists  $\nu > \bar{\nu}$  such that  $A_\nu \subset W$ .

Proof. Suppose not. Then there exists  $\bar{\nu}$  such that for all  $\nu > \bar{\nu}$   $A_\nu \not\subset W$ , i.e.,  $A_\nu \cap W^c \neq \emptyset$ . Pick  $x^\nu \in A_\nu \cap W^c$  and manufacture a subnet  $\{x^\nu\}$ . Since this subnet lies in  $K$ , there exists a further subnet  $\{x^{\nu\alpha}\}$  and  $\bar{x} \in K$  such that  $x^{\nu\alpha} \rightarrow \bar{x}$ . Since  $x^{\nu\alpha} \in W^c$  for all  $\alpha$  and since  $W^c$  is closed,  $\bar{x} \in W^c$ . Furthermore,  $\bar{x} \in \limsup A_\nu$ . Hence

$$\limsup A_\nu \cap W^c \neq \emptyset,$$

a contradiction which proves the lemma.

Lemma 2: For any normal topological space X, let  $\{\tau_\nu\}$  be a net chosen from  $\mathcal{M}_+^1(X, \tau)$ ,  $\tau_\nu \rightarrow \tau$  and  $\{A_\nu\}$  a net of subsets of a compact set K such that  $\tau_\nu(A_\nu) \geq m$  for all  $\nu$  and for some  $m > 0$ . Then  $\tau(\limsup A_\nu) \geq m$ .

Proof. Denote  $\limsup A_\nu$  by  $\bar{A}$ . Suppose  $\tau(\bar{A}) < m$ , i.e., there exists  $\delta > 0$  such that  $\tau(\bar{A}) < m - \delta$ . This implies that  $\tau(\bar{A}^c) > 1 - m + \delta$ . Since  $\tau \in \mathcal{M}_+^1(X, \tau)$ , there exists a compact set L such that

$$\tau(L) > 1 - m + (\delta/2).$$

Since X is normal, there exist open sets V and W such that  $V \cup W = \phi$  and  $\bar{A} \subset W$  and  $L \subset V$ .

By monotonicity of measures,  $\tau(V) \geq \tau(L) > 1 - m + (\delta/2)$ . Hence  $\tau(V^c) \leq m - (\delta/2)$ . Since  $V^c$  is a closed set, we can appeal to Theorem 2.4 to assert that  $\limsup \tau_\nu(V^c) \leq \tau(V^c)$ . Hence there exists a subnet  $\{\tau_\rho\}$  and  $\bar{\rho}$  such that for all  $\rho > \bar{\rho}$ ,

$$\tau_\rho(V^c) \leq \tau(V^c) + (\delta/4).$$

Since  $\limsup A_\rho \subset \bar{A} \subset W$ , we can appeal to Lemma 1 to find a  $\rho' > \bar{\rho}$  such that  $A_{\rho'} \subset W$ . Hence  $A_{\rho'} \subset V^c$ . Again by monotonicity of measures,  $\tau_{\rho'}(A_{\rho'}) \leq m - (\delta/4)$ , a contradiction which proves the lemma.

Lemma 3.  $\mathcal{C}_A \times A$  is a normal space.

Proof: A is compact Hausdorff, and hence  $\sigma$ -compact and regular Hausdorff. Since  $\mathcal{C}_A$  is a metric space, we can appeal to a result of Michael [see Morita (1964, p. 25, Assertion d)] to complete the proof of the lemma.

Claim 8.  $Q$  is an upper hemicontinuous correspondence.

Since  $\mathcal{T}$  is compact, it suffices to show that  $Q$  has a closed graph [see, Berge (1966, p. 112, Corollary)]. Towards this end, let  $\tau^v \rightarrow \tau$ ,  $\rho^v \in Q(\tau^v)$ ,  $\rho^v \rightarrow \rho$ . Assume that  $\rho \notin Q(\tau)$ , i.e.,  $\rho(B_\tau) < 1$ . Hence there exists  $\varepsilon > 0$  such that  $\rho(B_\tau) < 1 - \varepsilon$ .

Since  $\mu$  is a Radon probability on  $\mathcal{U}_A$ , there exists a compact subset of  $\mathcal{U}_A$  such that  $\mu(M) > 1 - (\varepsilon/4)$ . Since  $\{\rho^v\}$  and  $\rho$  are in (recall that  $\mathcal{T}$  is compact and hence closed), certainly

$$\rho^v(AxM) = \mu(M) = 1 - (\varepsilon/4) \text{ for all } v,$$

$$\rho(AxM) = \mu(M) = 1 - (\varepsilon/4).$$

Since  $A$  is compact,  $M \times A$  is a compact set in the product topology.

Call it  $K$ . Now let

$$B_{\tau^v}^K = B_{\tau^v} \cap K \text{ for all } v.$$

We claim that  $\rho^v(B_{\tau^v}^K) > 1 - (\varepsilon/2)$  for each  $v$ . Suppose not. Then for some  $v$ ,  $\rho^v(B_{\tau^v}^K) \leq 1 - (\varepsilon/2)$ . But  $\rho^v(B_{\tau^v}) = 1$  by hypothesis. Hence there exists a measurable subset  $Q$  of  $B_{\tau^v}$  such that  $Q \cap K = \emptyset$  and  $\rho^v(Q) \geq (\varepsilon/2)$ . But this contradicts the fact that  $\rho^v(K) = 1 - (\varepsilon/4)$ .

By virtue of Lemma 3, we can appeal to Lemma 2 to conclude that

$$\rho(\limsup B_{\tau^v}^K) \geq 1 - (\varepsilon/2)$$

We can now appeal to Claim 4 to assert that

$$\limsup(B_{\tau^v}^K) \subset B_\tau^K.$$



Since  $B_\tau^K \subset B_\tau$ , monotonicity of a measure yields

$$\rho(B_\tau) \geq 1 - (\varepsilon/2)$$

a contradiction which completes the proof of the claim.

We can now apply the Ky Fan fixed point theorem [see, for example, Berge (1963, p. 251)] to the map  $Q$  to complete the proof of the theorem.

Proof of Corollary. This is a direct consequence of Theorem 2.9 and Theorem 3.1.

#### Proof of Theorem 3.2

As in Claim 8, it suffices to show that  $\Gamma$  has a closed graph. Towards that end, let  $\mu^\nu \rightarrow \mu$ ,  $\tau^\nu \in \Gamma(\mu^\nu)$ ,  $\tau^\nu \rightarrow \tau$ . We have to show that

$$(i) \quad \tau \ll_A \mu,$$

$$(ii) \quad \tau(B_\tau) = 1$$

(i) is a consequence of Theorem 2.5. To show (ii) we repeat the arguments of the proof of Claim 8 with  $\{\tau^\nu\}$  substituted for the net  $\{\rho^\nu\}$ .

Proof of Theorem 3.3: The first claim is a direct consequence of Theorem 2.8. The second claim then follows from Theorem 3.1.

#### Proof of Theorem 4.1

Since  $A$  is Hausdorff compact by hypothesis and  $\mathcal{M}_+^1(A, t)$  is Hausdorff compact by virtue of Theorems 2.2 and 2.3,  $\text{Ax} \mathcal{M}_+^1(A, t)$  is a Hausdorff

compact space. We can now appeal to Corollary 4.3 in Dolecki et al. (1983) to complete the proof.

#### Proof of Theorem 4.2

The reader can check that it is only the proof of Theorem 3.1 that does not routinely extend to the space  $\mathcal{U}_A^h$  and we shall confine ourselves to it.

In the proof of Theorem 3.1, the space  $\mathcal{U}_A$  is essentially involved in Claims 4, 5, 6 and 8. We begin with Claim 4 under which we have to show

B:  $\mathcal{I} \rightarrow 2^{Ax \mathcal{U}_A^h}$  has a closed graph.

Let  $\tau^v \rightarrow \tau$ ,  $(u^v, a^v) \rightarrow (u, a)$  and  $(u^v, a^v) \in B_{\tau^v}$ . Suppose  $(u, a) \notin B_{\tau}$ . Then there exists  $x \in A$  such that  $u(x, \tau_A) > u(a, \tau_A)$ , i.e., there exists  $\varepsilon > 0$  such that

$$(*) \quad u(x, \tau_A) > u(a, \tau_A) + \varepsilon.$$

Since  $u^v \rightarrow u$ , hypograph  $u^v \rightarrow$  hypograph  $u$  in the topology of closed convergence. Since  $(a, \tau_A, u(a, \tau_A)) \in \text{hypograph } u$ , and since by virtue of Theorem 2.5,  $\tau_A^v \rightarrow \tau_A$ , for any neighborhood  $V$  of  $\tau_A$ , and any neighborhood  $W$  of  $a$ , there exists  $\bar{v}$  such that

$$(\text{hypograph } u^v) \cap WxVx B_{\varepsilon/2}(u(a, \tau_A)) \neq \emptyset \quad \text{for all } v > \bar{v}.$$

Now let  $\rho > \bar{v}$  and consider  $u^\rho(a^\rho, \tau_A^\rho)$ . Certainly,

$$(**) \quad |u^\rho(a^\rho, \tau_A^\rho) - u(a, \tau_A)| < \varepsilon/2.$$

On substituting (\*\*) in (\*), we contradict the fact that  $(u^p, a^p) \in B_{\tau^p}$ . The proof of the Claim is finished.

The proof of Claim 5 is now routine. The proofs of Claim 6 and 8 rely on the normality of the space  $\mathcal{U}_A \times A$ . In particular, Lemma 3 used the fact that  $\mathcal{U}_A$  is a metric space. However, Theorem 4.1 yields that  $\mathcal{U}_A^h$  is Hausdorff compact and hence  $\mathcal{U}_A^h \times A$  is Hausdorff compact and hence normal; see, for example, Willard (1970, Theorem 17.10, p. 121). Hence the proofs of Claims 6 and 8 can be completed along identical lines.

The strengthening of Theorem 3.2 follows from the observation that a mapping with a closed graph is upper hemicontinuous if the range space is compact; see, for example, Berge (1963, p. 251). However, the range space is compact by virtue of Theorems 2.3 and 4.1.

#### Proof of Theorem 4.3

The proof of Theorem 4.3 is essentially the observation that on the space of continuous functions on a compact set, the  $\tau_e$  topology is finer than the sup-norm topology. However, for this observation see Dolecki et al. (1983, p. 427, paragraph 6).

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